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A SLOPE MODIFICATION METHOD FOR SHALLOW WATER EQUATIONS

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SUMMARY

A slope modification method is proposed for non-oscillatory schemes based on the Lax–Friedrich solver. The modified scheme is proved to be total-variation-diminishing (TVD) and second-order accurate. Application of the scheme to the shallow water equations produces sharp profiles for shocks and achieves high accuracy in the smooth regions of the solution.

KEY WORDS Slope modification method Non-oscillatory schemes Shallow water equations

1. INTRODUCTION

In this paper we consider numerical approximations to weak solutions of hyperbolic conservation laws of the form

$$u_t+[f(u)]_x=0, \quad -\infty<x<\infty, \quad t>0, \tag{1a}$$

with the initial condition

$$u(x, 0)=u_0(x). \tag{1b}$$

The solution of the above problem may develop discontinuities even though $u_0(x)$ is a smooth function. In recent years much interest has been shown in developing numerical schemes of higher accuracy which give sharp profiles for the discontinuities.

Harten[1] introduced high-resolution total-variation-diminishing (TVD) schemes and many others[2,3] made further contributions to the development of them.

If v_j^n denotes the numerical approximation to problem (1) at (x_j, t_n), $x_j=j\Delta x$, $t_n=n\Delta t$, then the finite difference scheme in conservation form approximating (1) can be written as

$$v_j^{n+1}=v_j^n-\lambda(f_{j+1/2}-f_{j-1/2}), \tag{2a}$$

where $\lambda=\Delta t/\Delta x$ is the CFL number and $f_{j+1/2}$ is the numerical flux function of $2k$ variables,

$$f_{j+1/2}=f(v_{j-k+1}^n, \ldots, v_{j+k}^n), \tag{2b}$$

which is consistent with (1) in the sense that

$$f(v, v, \ldots, v)=f(v). \tag{2c}$$

The total variation of any grid function $\{v_j^n\}$ is defined to be

$$\mathrm{TV}(v^n)=\sum_{j=-\infty}^{\infty}|v_{j+1}^n-v_j^n|. \tag{3a}$$

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A scheme in the form (2a) is said to be TVD if it satisfies at each time step

$$\mathrm{TV}(v^{n+1}) \leqslant \mathrm{TV}(v^n). \tag{3b}$$

Recently, Nessyahu and Tadmor[4] have constructed high-resolution central difference schemes which are proved to be TVD. The Lax–Friedrich solver is used as the building block. The simplicity of the schemes lies mainly in the fact that they do not require solving any Riemann problems. An artificial compression method for non-oscillatory schemes has been developed by Yang[5] which greatly improves the resolution of the discontinuities.

The main aim of this paper is to incorporate the work of Nessyahu and Tadmor[4] andYang[5] to obtain a more accurate TVD scheme which will sharply resolve the discontinuities and will also be simple to implement. In Section 2 we describe the high-resolution central difference schemes. In Section 3 we present the slope modification method. Section 4 extends the method to systems of conservation laws. Section 5 gives the application of the method to the shallow water equations and a discussion of the results. Conclusions are drawn in Section 6.

2. HIGH-RESOLUTION CENTRAL DIFFERENCE SCHEMES

The staggered form of the Lax–Friedrich scheme[4] approximating (1) is given by

$$v_{j+1/2}(t+\Delta t) = \tfrac{1}{2}(v_j + v_{j+1}) - \lambda\{f[v_{j+1}(t)] - f[v_j(t)]\}. \tag{4}$$

It is equivalent to a recipe for solving a sequence of Riemann problems by approximating the solution at time level t by piecewise constants over cells of width $\Delta x = x_{j+1/2} - x_{j-1/2}$ in the form

$$\bar{v}(x,t) = v_j(t), \quad x_{j-1/2} \leqslant x < x_{j+1/2}. \tag{5}$$

At time level $t+\Delta t$

$$v(x, t+\Delta t) = R\left(\frac{x - x_{j+1/2}}{\Delta t}, v_j, v_{j+1}\right), \quad x_j \leqslant x < x_{j+1}, \tag{6}$$

where R is the Riemann solver. When the solution is projected back on the space of piecewise-constant grid functions and integrated over a staggered grid, we also get

$$v_{j+1/2}(t+\Delta t) = \bar{v}(x, t+\Delta t) \tag{7a}$$

$$= \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} v(y, t+\Delta t)\,\mathrm{d}y \quad (x_j \leqslant x < x_{j+1}) \tag{7b}$$

$$= \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} R\left(\frac{x - x_{j+1/2}}{\Delta t}, v_j, v_{j+1}\right)\mathrm{d}x. \tag{7c}$$

The disadvantage of scheme (4) is that it has excessive numerical viscosity. In order to compensate this, it is proposed[4] to formulate the Riemann problems by approximating the solution at time t with piecewise-linear functions as follows.

$$L_j(x,t) = v_j(t) + \frac{1}{\Delta x}(x - x_j)v_j', \quad x_{j-1/2} \leqslant x < x_{j+1/2} \tag{8}$$

Then

$$v_{j+1/2}(t+\Delta t) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \mathrm{GR}(x, t+\Delta t, L_j, L_{j+1})\,\mathrm{d}x, \tag{9}$$

where GR represents the generalized Riemann solver. The second-order accuracy is governed by the following choice of the numerical derivative $v'_j/\Delta x$:

$$\frac{1}{\Delta x} v'_j = \frac{\partial}{\partial x} v(x=x_j, t) + O(\Delta x). \tag{10}$$

Integrating (9) over the staggered grid using the midpoint rule, we obtain

$$v_{j+1/2}(t+\Delta t) = \tfrac{1}{2}[v_j(t)+v_{j+1}(t)] + \tfrac{1}{8}(v'_j - v'_{j+1}) - \lambda\{f[v(x_{j+1}, t+\Delta t/2)] - f[v(x_j, t+\Delta t/2)]\}. \tag{11a}$$

Here $v(x_j, t+\Delta t/2)$ may be approximated by Taylor expansion as

$$v(x_j, t+\Delta t/2) = v_j(t) - \tfrac{1}{2}\lambda f'_j, \tag{11b}$$

where f'_j is the numerical flux derivative satisfying a condition of the form (10). Thus the scheme takes the form

$$v_j(t+\Delta t/2) = v_j(t) - \tfrac{1}{2}\lambda f'_j, \tag{12a}$$

$$v_{j+1/2}(t+\Delta t) = \tfrac{1}{2}[v_j(t)+v_{j+1}(t)] - \lambda(g_{j+1} - g_j), \tag{12b}$$

$$g_j = f[v_j(t+\Delta t/2)] + (1/8\lambda)v'_j. \tag{12c}$$

3. SLOPE MODIFICATION METHOD

The spurious oscillations in the numerical solutions are reduced if (12) is made to satisfy the TVD property. The numerical derivatives are chosen[4] as

$$0 \leqslant v'_j \operatorname{sgn}(x) \leqslant \text{const}_v |\text{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\}|, \tag{13a}$$

where $\Delta v_{j+1/2} = v_{j+1} - v_j$ and

$$\text{MM}\{x, y\} = \tfrac{1}{2}[\operatorname{sgn}(x) + \operatorname{sgn}(y)]\min\{|x|, |y|\}. \tag{13b}$$

The numerical flux derivatives are chosen similarly so that scheme (12) is TVD.

A modification is suggested in the choice of the slopes v'_j and f'_j which enables the scheme to achieve high resolution of discontinuities while maintaining the TVD nature. On the basis of the work of Yang[5] we choose v'_j as

$$v'_j = \text{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\} + 2\,\text{MM}\{\mu_j \text{MM}\{\delta_{j+1/2}, \delta_{j-1/2}\}, \text{MM}\{\beta_{j+1/2}, \beta_{j-1/2}\}\}, \tag{14}$$

where μ_j are positive constants and

$$\delta_{j-1/2} = \Delta v_{j-1/2} - \tfrac{1}{2}(\text{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\} + \text{MM}\{\Delta v_{j-1/2}, \Delta v_{j-3/2}\}),$$

$$\beta_{j-1/2} = \Delta v_{j-1/2} - \tfrac{1}{2}\text{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\},$$

$$\beta_{j+1/2} = \Delta v_{j+1/2} - \tfrac{1}{2}\text{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\}.$$

$\delta_{j+1/2}$ follows from $\delta_{j-1/2}$ by replacing j by $j+1$. f'_j is chosen as

$$f'_j = a(v_j)v'_j, \tag{15}$$

where $a(v_j) = (\partial f/\partial u)|_{x=x_j}$.

Lemma

Scheme (12) is TVD if the modified numerical flux g_j given by (12c) satisfies the generalized CFL condition

$$\lambda \left| \frac{\Delta g_{j+1/2}}{\Delta v_{j+1/2}} \right| \leqslant \frac{1}{2}. \tag{16}$$

Proof

$$v_{j+1/2}(t+\Delta t) - v_{j-1/2}(t+\Delta t) = \Delta v_{j+1/2}\left(\frac{1}{2} - \lambda \frac{\Delta g_{j+1/2}}{\Delta v_{j+1/2}}\right) + \Delta v_{j-1/2}\left(\frac{1}{2} + \lambda \frac{\Delta g_{j-1/2}}{\Delta v_{j-1/2}}\right).$$

From (16) it follows that the terms in parentheses on the RHS are positive and TVD follows from

$$\mathrm{TV}[v(t+\Delta t)] = \sum_j |v_{j+1/2}(t+\Delta t) - v_{j-1/2}(t+\Delta t)| \leqslant \mathrm{TV}[v(t)].$$

Theorem

Let v'_j and f'_j be chosen as in (14) and (15) respectively. Let

$$\lambda \max |a(v_j)| \leqslant \eta \tag{17a}$$

with

$$\eta \leqslant \frac{1}{2(1+2\mu_j)} \{[4 + 4(1+2\mu_j) - (1+2\mu_j)^2]^{1/2} - 2\} \tag{17b}$$

hold. Then scheme (12) is TVD.

Proof. It is sufficient to prove that (16) holds.

$$\left| \lambda \frac{\Delta g_{j+1/2}}{\Delta v_{j+1/2}} \right| \leqslant \lambda \left| \frac{f[v_{j+1}(t+\Delta t/2)] - f[v_j(t+\Delta t/2)]}{\Delta v_{j+1/2}} \right| + \frac{1}{8} \left| \frac{\Delta v'_{j+1/2}}{\Delta v_{j+1/2}} \right|.$$

Introducing the term $v_{j+1}(t+\Delta t/2) - v_j(t+\Delta t/2)$ on the RHS and using (17a) and (11b), we get

$$\lambda \frac{\Delta g_{j+1/2}}{\Delta v_{j+1/2}} \leqslant \eta \left(1 + \frac{\lambda}{2} \left| \frac{\Delta f'_{j+1/2}}{\Delta v_{j+1/2}} \right| \right) + \frac{1}{8} \left| \frac{\Delta v'_{j+1/2}}{\Delta v_{j+1/2}} \right|. \tag{18a}$$

Now

$$\frac{\Delta v'_{j+1/2}}{\Delta v_{j+1/2}} \leqslant \max \left\{ \left| \frac{v'_j}{\Delta v_{j+1/2}} \right|, \left| \frac{v'_{j+1}}{\Delta v_{j+1/2}} \right| \right\} \leqslant 1 + 2\mu_j, \tag{18b}$$

$$\left| \frac{\Delta f'_{j+1/2}}{\Delta v_{j+1/2}} \right| \leqslant \max \left\{ \left| \frac{f'_j}{\Delta v_{j+1/2}} \right|, \left| \frac{f'_{j+1}}{\Delta v_{j+1/2}} \right| \right\}. \tag{18c}$$

Using (15), (17a) and (18b), we get

$$\left| \frac{\Delta f'_{j+1/2}}{\Delta v_{j+1/2}} \right| \leqslant \frac{\eta}{\lambda} (1 + 2\mu_j). \tag{18d}$$

With the above bounds (18a) reduces to

$$\left|\lambda \frac{\Delta g_{j+1/2}}{\Delta v_{j+1/2}}\right| \leqslant \eta[1+(\eta/2)(1+2\mu_j)]+\tfrac{1}{8}(1+2\mu_j), \tag{18e}$$

where

$$\eta[1+(\eta/2)(1+2\mu_j)]+\tfrac{1}{8}(1+2\mu_j)\leqslant\tfrac{1}{2} \tag{18f}$$

has the solution (17b); hence the proof.

Remarks

1. The range of μ_j for which $\eta>0$ is given by $0\leqslant\mu_j<1{\cdot}5$.
2. When $\mu_j=0$, $v'_j=\mathrm{MM}\{\Delta v_{j+1/2}, \Delta v_{j-1/2}\}$, which corresponds to the case $\mathrm{const}_v=1$ in the choice of v'^{4}_j.

4. EXTENSION TO SYSTEM OF CONSERVATION LAWS

In this section we consider the extension of the method described in Sections 2 and 3 to the system of hyperbolic conservation laws

$$\mathbf{U}_t+[\mathbf{F}(\mathbf{U})]_x=0, \quad -\infty<x<\infty, \quad t>0, \tag{19a}$$

with the Riemann initial conditions

$$\mathbf{U}(x,0)=\begin{cases}\mathbf{U}_\mathrm{L}, & x\leqslant x_0,\\ \mathbf{U}_\mathrm{R}, & x>x_0.\end{cases} \tag{19b}$$

Here $\mathbf{U}(x,t)$ is the N-vector

$$\mathbf{U}=[u_1(x,t),\ldots,u_N(x,t)]^\mathrm{T}$$

and $\mathbf{F}(\mathbf{U})$ is the flux vector

$$\mathbf{F}(\mathbf{U})=[f_1(\mathbf{U}),\ldots,f_N(\mathbf{U})]^\mathrm{T}.$$

System (19a) can also be written as

$$\mathbf{U}_t+\mathbf{A}(\mathbf{U})\mathbf{U}_x=0, \tag{20a}$$

where $\mathbf{A}(\mathbf{U})$ is the Jacobian matrix given by

$$A_{m,l}(\mathbf{U})=\frac{\partial f_m}{\partial u_l}, \quad m,l=1,2,\ldots,N. \tag{20b}$$

Let the numerical approximation to system (19) at the grid point x_j be given by the vector

$$\mathbf{U}_j=(u_{j,1},\ldots,u_{j,N})^\mathrm{T}.$$

Let $\mathbf{U}_{j+1/2}$ denote some average of $\mathbf{U}_j$ and $\mathbf{U}_{j+1}$. A particular average due to Roe[6] can be considered and the averaged Jacobian $\mathbf{A}_{j+1/2}=\mathbf{A}(\mathbf{U}_j,\mathbf{U}_{j+1})$ satisfying

$$\mathbf{F}(\mathbf{U}_{j+1})-\mathbf{F}(\mathbf{U}_j)=\mathbf{A}_{j+1/2}(\mathbf{U}_{j+1}-\mathbf{U}_j)$$

can be used to incorporate the characteristic information into the scheme, which improves the resolution.

Let $a^k_{j+1/2}$ denote the eigenvalues of $\mathbf{A}_{j+1/2}$ and $\mathbf{R}^k_{j+1/2}$ ($\mathbf{L}^k_{j+1/2}$) denote the corresponding right (left) eigenvectors for $k=1, \ldots, N$. The vector $\Delta\mathbf{U}_{j+1/2}$ can be written as a linear combination of $\mathbf{R}^k_{j+1/2}$ as

$$\Delta\mathbf{U}_{j+1/2}=\sum_k \alpha^k_{j+1/2}\mathbf{R}^k_{j+1/2}, \tag{21a}$$

where

$$\alpha^k_{j+1/2}=\mathbf{L}^k_{j+1/2}\Delta\mathbf{U}_{j+1/2}, \quad k=1, 2, \ldots, N. \tag{21b}$$

Following (21), the characteristic-wise choice of the numerical derivatives can be accomplished analogously to (14) and (15) as

$$\mathbf{U}'_j=\sum_k (\mathrm{MM}\{\alpha^k_{j+1/2}, \alpha^k_{j-1/2}\}+2\,\mathrm{MM}\{\mu_j \mathrm{MM}\{\theta^k_{j+1/2}, \theta^k_{j-1/2}\}, \mathrm{MM}\{\nu^k_{j+1/2}, \nu^k_{j-1/2}\}\})\mathbf{R}^k_j, \tag{22}$$

where

$$\theta^k_{j-1/2}=\alpha^k_{j-1/2}-\tfrac{1}{2}(\mathrm{MM}\{\alpha^k_{j+1/2}, \alpha^k_{j-1/2}\}+\mathrm{MM}\{\alpha^k_{j-1/2}, \alpha^k_{j-3/2}\}),$$
$$\nu^k_{j-1/2}=\alpha^k_{j-1/2}-\tfrac{1}{2}\mathrm{MM}\{\alpha^k_{j+1/2}, \alpha^k_{j-1/2}\},$$
$$\nu^k_{j+1/2}=\alpha^k_{j+1/2}-\tfrac{1}{2}\mathrm{MM}\{\alpha^k_{j+1/2}, \alpha^k_{j-1/2}\}.$$

$\theta^k_{j+1/2}$ is obtained by replacing j by $j+1$ in $\theta^k_{j-1/2}$ and

$$\mathbf{F}'_j=\mathbf{A}_{j+1/2}\mathbf{U}'_j. \tag{23}$$

Equipped with the vector of numerical derivatives $\mathbf{U}'_j$ and $\mathbf{F}'_j$, scheme (12) can be extended to the system of equations as

$$\mathbf{U}_j(t+\Delta t/2)=\mathbf{U}_j(t)-\tfrac{1}{2}\lambda\mathbf{F}'_j, \tag{24a}$$

$$\mathbf{U}_{j+1/2}(t+\Delta t)=\tfrac{1}{2}[\mathbf{U}_j(t)+\mathbf{U}_{j+1}(t)]-\lambda(\mathbf{G}_{j+1}-\mathbf{G}_j), \tag{24b}$$

$$\mathbf{G}_j=\mathbf{F}[\mathbf{U}_j(t+\Delta t/2)]+(1/8\lambda)\mathbf{U}'_j. \tag{24c}$$

5. APPLICATION TO SHALLOW WATER EQUATIONS

The shallow water equations in one-dimension are given in the conservation form

$$\begin{bmatrix} S \\ Q \end{bmatrix}_t+\begin{bmatrix} Q \\ Q^2/S+P(S) \end{bmatrix}_x=0, \tag{25}$$

where S is the cross-section of the flow, $Q=SV$ is the momentum along the X-direction, V is the velocity of the flow and $P(S)=S^2$ is the pressure effect.

The Jacobian matrix for system (25),

$$\mathbf{A}(\mathbf{U})=\begin{bmatrix} 0 & 1 \\ 2S-V^2 & 2V \end{bmatrix}, \tag{26a}$$

has the eigenvalues $(a^1, a^2)=(V-C, V+C)$, where C denotes the local speed given by $\sqrt{(2S)}$. The right eigenvectors

$$\mathbf{R}^1=\begin{bmatrix} 1 \\ V-C \end{bmatrix} \quad \text{and} \quad \mathbf{R}^2=\begin{bmatrix} 1 \\ V+C \end{bmatrix} \tag{26b}$$

form the matrix $\mathbf{R}=[\mathbf{R}^1, \mathbf{R}^2]$, which has the inverse

$$\mathbf{R}^{-1}=\frac{1}{2C}\begin{bmatrix} V+C & -1 \\ C-V & 1 \end{bmatrix}. \tag{26c}$$

The left eigenvectors are the rows of $\mathbf{R}^{-1}$. Roe's average for (25) is given by

$$v_{j+1/2}=\frac{S_{j+1}^{1/2}V_{j+1}+S_j^{1/2}V_j}{S_{j+1}^{1/2}+S_j^{1/2}}, \tag{27a}$$

$$S_{j+1/2}=\frac{S_{j+1}+S_j}{2}. \tag{27b}$$

From (21c) we get

$$\begin{bmatrix} \alpha_{j+1/2}^1 \\ \alpha_{j+1/2}^2 \end{bmatrix}=\frac{1}{2C_{j+1/2}}\begin{bmatrix} (V+C)_{j+1/2}\,\Delta S_{j+1/2}-\Delta Q_{j+1/2} \\ (C-V)_{j+1/2}\,\Delta S_{j+1/2}+\Delta Q_{j+1/2} \end{bmatrix}. \tag{28}$$

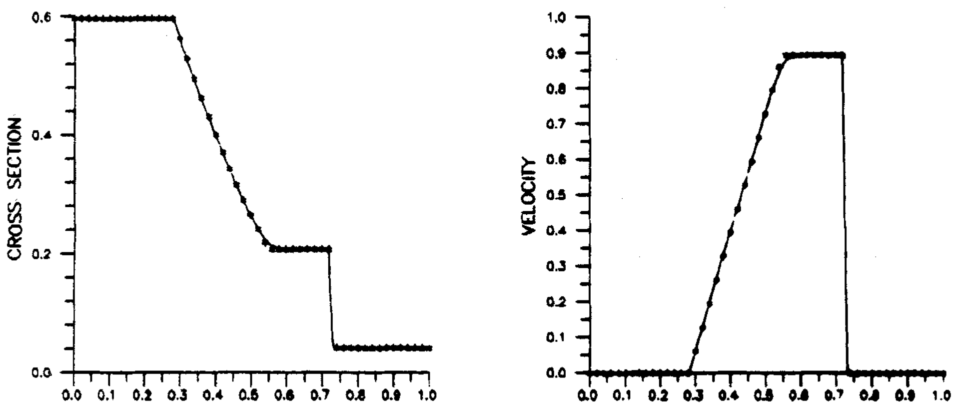

Figure 1. Slope modification method

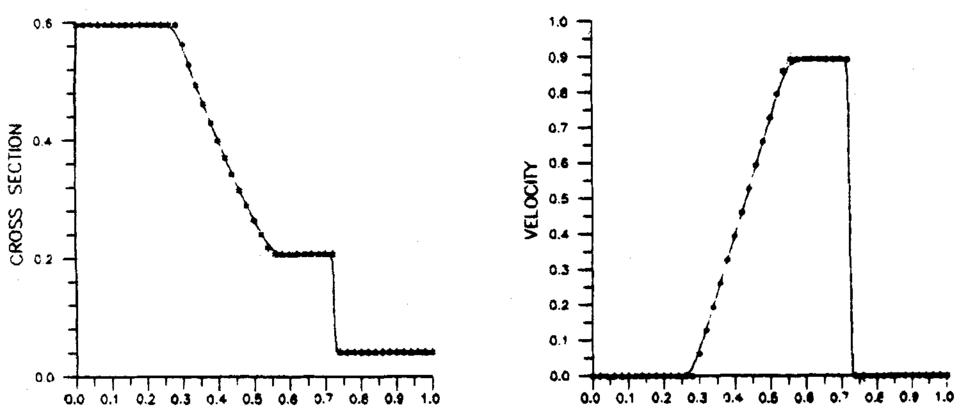

Figure 2. Nessyahu and Tadmor's scheme

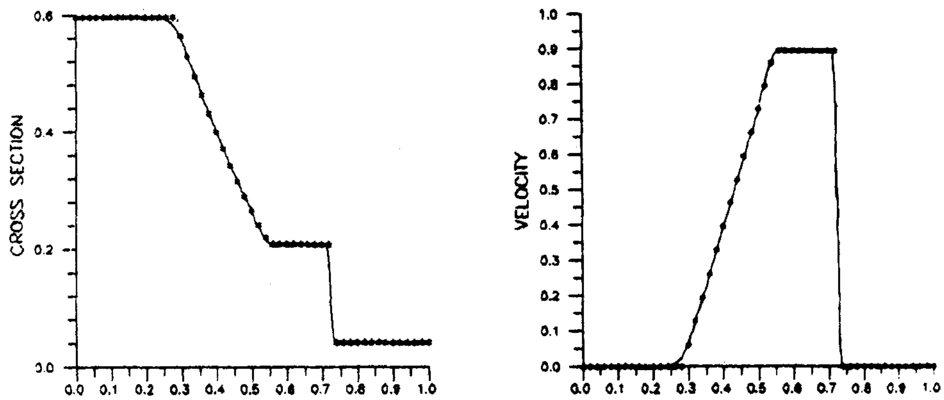

Figure 3. Yee's scheme

Numerical results

Solutions are obtained using scheme (24) for the shallow water equations (25). The initial conditions are taken as

$$S_L = 0{\cdot}597, \qquad S_R = 0{\cdot}04166, \qquad Q_L = 0, \qquad Q_R = 0,$$

for $0 \leqslant x \leqslant 1$ with $x_0 = 0{\cdot}5$. The constants μ_j are chosen equal to 1·45 for all j. Results are obtained at time $t = 0{\cdot}2$ using a grid of 200 points along the X-direction. For comparison we have also computed the same problem using two other TVD schemes due to Yee[3] and Nessyahu and Tadmor.[4] The latter scheme corresponds to our scheme (24) with $\mu_j = 0$ for all j. The results are shown in the form of graphs (Figures 1–3). The exact solutions are indicated by symbols, the approximate solutions by full lines.

6. CONCLUSIONS

From the numerical results it is noted that the slope modification method produces sharp profiles for the discontinuities as compared to the other methods. While the schemes due to Yee[3] and Nessyahu and Tadmor[4] fail to locate the shock exactly, the results for scheme (24) are indistinguishable from the exact solution. The computational complexity is comparable with that of the other two schemes, since all of them are free of Riemann solvers.

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